STP 83-48

The Persistence of Boson Condensation in the van der Waal's Limit

J.T. Lewis

School of Theoretical Physics
Dublin Institute for Advanced Studies
10 Burlington Road
Dublin 4, Ireland

J.V. Pulè

Department of Mathematical Physics
University College
Belfield
Dublin 4, Ireland

Ph. de Smedt

Institute of Theoretical Physics
Katholieke Universiteit Leuven
Celestijnenlaan 200 D
B-3030 Leuven
Belgium

1. Introduction.

Consider a gas of Bose particles interacting via a pair potential $V_\lambda(x) = q(x) + \lambda^\nu U(\lambda x)$ (ν being the dimension of the system). The van der Waal's limit consists in considering the thermodynamic limit of this system, followed by letting λ tend to zero. Physically, this means considering a sequence of systems with ever increasing range of interactions, but still the range of the interaction is always vanishing small compared to the size of the system.

Kac, Uhlenbeck and Hemmer [1] were the first to calculate this limit rigourously for a one-dimensional classical gas with a two-body interaction potential of the form

$$V_\lambda(x) = q(|x|) + \tfrac{1}{2}\alpha\lambda\exp(-\lambda|x|) \quad , \quad \alpha < 0 \, , \, \lambda > 0 \, ,$$

where
$$q(s) = \begin{cases} +\infty & , \; s \leq a \, , \\ 0 & , \; s > a \, . \end{cases}$$

Their results were later generalized by Lebowitz and Penrose [2] to classical systems in any dimensions and with two-body interactions $V_\lambda(x) = q(x) + \lambda^\nu U(\lambda x)$, where the short-range term $q(x)$ has a hard core part, but is, as $U(x)$, for the rest quite arbitrary. Their main result is the following :

$$f_{v.d.w.}(\rho) = \text{C.E.}\left\{ \tfrac{1}{2} a\rho^2 + f(\rho) \right\} ,$$

where

i) f v.d.w. (ρ) is the free energy of the system in the van der Waal's limit.

ii) $f(\rho)$ is the limiting free energy of the system without the long range part.

iii) $a = \int d^\nu x \, U(x)$.

iv) C.E. stands for convex envelope (i.e. the greatest convex function which is everywhere less than or equal to $\tfrac{1}{2}a\rho^2 + f(\rho)$).

If $a \geq 0$, this implies that $f_{v.d.w.} = \tfrac{1}{2}a\rho^2 + f(\rho)$, where the term $\frac{1}{2}\alpha\rho^2$ is typical for a mean field interaction. The preceding results were later generalized by Lieb [3] to quantum mechanical systems with any statistics. He derived certain results which allow us to carry through the Lebowitz-Penrose analysis for quantum systems. In this article, we present an alternative derivation of the van der Waal's limit for Boson systems. Although our analysis is restricted to pair-potentials of the type $\lambda^\nu U(\lambda x)$ where U (x) is a positive-definite function, it has some interesting advantages.

1) It shows quite clearly that the mean field system can be viewed as a limit of systems with very weak interactions, but which are extremely long range. It suggests therefore a natural way to do perturbation theory of interacting models around the mean field model, which is a sounder system to perturb around than the free model, which is full of pathologies.
2) Our method is essentially easy and straightforward.
3) It allows us to prove some result about energy-level occupation densities. Most importantly, it allows us to prove that generalized condensation [4] shows up in the van der Waal's limit when the density is large enough. These last results are new and, as we will explain later on, by no means trivial.

2. Description of the model and notations.

Consider a system of N Bosons in a ν-dimensional box Λ^L described by the following Hamiltonian:

$$H^L_{int,\lambda} = T^L + \Phi^L_\lambda ,$$

where Λ^L are the cubes $\left[-\frac{L}{2}, \frac{L}{2}\right]^\nu$, T^L is the kinetic energy-term derived from the one-particle Hamiltonian $h^L = -\frac{1}{2}\Delta$ with periodic boundary conditions on Λ_L. Denote the eigenvalues of h^L by $E^L_1 = 0 < E^L_2 \leqslant E^L_3 \leqslant \cdots$ and its corresponding eigenvectors by $\phi^L_k(x)$. (Note: ϕ^L_k is of the form $C^L_k \exp(i\xi\cdot x)$ where $\xi = \left(\frac{2\pi n_1}{L}, \ldots, \frac{2\pi n_\nu}{L}\right)$, $n_j \in \mathbb{Z}$.)

We assume moreover that Φ^L_λ is of the form

$$\Phi^L_\lambda(x_1, \ldots, x_\nu) = \sum_{1\leq i<j\leq N} \lambda^\nu U(\lambda(x_i - x_j)) , \qquad (1)$$

where

i) U is in $L^1(\mathbb{R}^\nu)$. (2)

ii) is a function of positive type, i.e.

for all $n \in \mathbb{N}$, $c \in \mathbb{C}$, $x \in \mathbb{R}^\nu$: $\sum_{1 \leq i < j \leq n} \bar{c}_i c_j U(x_i - x_j) \geq 0$. (3)

iii) $a = \int d^\nu x \, U(x) > 0$. (4)

Consider also the mean field Hamiltonian

$$H_a^L = T^L + \frac{a}{2} \frac{(N^L)^2}{L^\nu} , \qquad \text{where } N^L \text{ is the number operator.} \tag{5}$$

(0; =0 corresponds to the free gas)

Define, as usual, the pressure $p_a^L(\mu)$ by

$$p_a^L(\mu) = \frac{1}{\beta L^\nu} \log \operatorname{Tr} \left[\exp\{-\beta(H_a^L - \mu N^L)\} \right]$$

and the KMS-state $\omega_{a,\mu}^L(\cdot)$ by

$$\omega_{a,\mu}^L(A) = \cancel{\frac{1}{\beta L^\nu}} \frac{\operatorname{Tr}[A \, e^{-\beta(H_a^L - \mu N^L)}]}{\operatorname{Tr}[e^{-\beta(H_a^L - \mu N^L)}]} .$$

Denote the corresponding quantities for the interacting Bose gas by $p_{int,\lambda}^L(\cdot)$ and $\omega_{int,\lambda,\mu}^L(\cdot)$.

Finally, we agree to the following :

$$\Phi^L \stackrel{def}{=} \Phi_1^L , \quad H_{int}^L \stackrel{def}{=} H_{int,1}^L , \quad p_{int}^L \stackrel{def}{=} p_{int,1}^L ; \quad \omega_{int,\mu}^L(\cdot) \stackrel{def}{=} \omega_{int,1,\mu}^L(\cdot) .$$

3. Some basic results about free and mean field Bose systems. [5]

a) The free gas: (see e.g. [4]).

Denote by $F^L(x)$ the function

$$F^L(u) = \frac{1}{L^\nu} \# \{ k : E_k^L \leq u \} . \tag{6}$$

Then $F^L(u) \longrightarrow F(u) = \dfrac{2u^{\nu/2}}{(2\pi)^{\nu/2}\,\nu\,\Gamma(\frac{\nu}{2})}$ when $L \longrightarrow \infty$. (6a)

It is well known that

i) $p_0^L(\mu)$ is defined only for $\mu < 0$.

ii) $$p_0(\mu) = \lim_{L\to\infty} p_0^L(\mu) = -\frac{1}{\beta}\int_{[0,\infty)} \log\left(1 - \exp\{-\beta(u-\mu)\}\right) du\,. \tag{7}$$

iii) $$\rho_0(\mu) \stackrel{def}{=} \lim_{L\to\infty} \rho_0^L(\mu) \stackrel{def}{=} \lim_{L\to\infty} \omega^L_{0,\mu}\left(\frac{N^L}{L^\nu}\right) = \int_{[0,\infty)} \frac{dF(u)}{e^{\beta(u-\mu)}-1}\,. \tag{8}$$

iv) The critical density, above which Bose-Einstein condensation occurs, is $$\rho_c = \lim_{\mu\uparrow 0} \rho_0(\mu) = \int_{[0,\infty)} \frac{dF(u)}{e^{\beta u}-1}\,. \tag{9}$$

Note that $\rho_c = \infty$ if $\nu \leqslant 2$.

b) <u>The mean field system:</u> (see e.g. [5], [6]).

The pressure $p_a^L(\mu)$ ($a>0$) is defined for all values of μ.

One has

i) $$p_a(\mu) = \lim_{L\to\infty} p_a^L(\mu) = p_0(\alpha) + \frac{(\mu-\alpha)^2}{2} \tag{10}$$

where α satisfies

$$\alpha = \begin{cases} \mu - a\rho_0(\alpha)\,, & \mu \leqslant \mu_c = a\rho_c\,, \\ 0\,, & \mu > \mu_c = a\rho_c\,. \end{cases} \tag{11}$$

ii) $$\rho_a(\mu) \stackrel{def}{=} \lim_{L\to\infty} \omega^L_{a,\mu}\left(\frac{N^L}{L^\nu}\right) = \frac{(\mu-\alpha)}{a}\,. \tag{12}$$

Now define the operators $N^L_{[c,d]}$ by:

$$X^L_{[c,d]} \stackrel{def}{=} \frac{1}{L^\nu} \sum_{\{k:\, E^L_k \in [c,d]\}} N^L_k \qquad \text{with} \quad 0 \leqslant c < d < \infty\,, \tag{13}$$

where N^L_k is the number operator for the k^{th} energy level.

Then

$$\lim_{L\to\infty} \omega^L_{a,\mu}\left(\exp\{\lambda X^L_{[c,d]}\}\right) = \exp\left\{\lambda \int_{[c,d]} \frac{dF(u)}{e^{\beta(u-\alpha)}-1}\right\}\,,\ c>0\,, \tag{14a}$$

$$\lim_{L\to\infty} \omega^L_{a,\mu} (\exp\{\lambda X^L_{[c,d]}\}) = \exp \lambda\{(\rho_a(\mu)-\rho_c)\theta(\rho_a(\mu)-\rho_c) + \int_{[c,d]} \frac{dF(u)}{e^{\beta(u-\alpha)}-1}\}, \tag{14b}$$

where α is as in (11) and where θ is the usual Heavyside function.

Moreover $\lim_{L\to\infty} \omega^L_{a,\mu} (\exp\{\lambda_1 X^L_{[c_1,d_1]} + \lambda_2 X^L_{[c_2,d_2]}\})$

$$= \lim_{L\to\infty} \omega^L_{a,\mu}(\exp\{\lambda_1 X^L_{[c_1,d_1]}\})\,\omega^L_{a,\mu}(\exp\{\lambda_2 X^L_{[c_2,d_2]}\}) . \tag{15}$$

The proof of equations (14) - (15) can be found in [6], but the idea for the proof is actually contained in [5]. As $X^L_{[c,d]} \leq \frac{N^L}{L^\nu}$ one easily verifies that (14) - (15) imply:

$$\lim_{L\to\infty} \omega^L_{a,\mu} (X^L_{[c_1,d_1]}) = \delta_{c,0} (\rho_a(\mu)-\rho_c)\theta(\rho_a(\mu)-\rho_c) + \int_{[c,d]} \frac{dF(u)}{e^{\beta(u-\alpha)}-1} \tag{16}$$

(where $\delta_{c,0} = \begin{cases} 1, & c=0, \\ 0, & c\neq 0, \end{cases}$)

and

$$\lim_{L\to\infty} \omega^L_{a,\mu} (X^L_{[c_1,d_1]} X^L_{[c_2,d_2]}) = \lim_{L\to\infty} \omega^L_{a,\mu} (X^L_{[c_1,d_1]}) \lim_{L\to\infty} \omega^L_{a,\mu} (X^L_{[c_2,d_2]}) . \tag{17}$$

Note that by virtue of (6a), $\int_{[c,d]} \frac{dF(u)}{e^{\beta(u-\alpha)}-1}$ can also be written as $(2\pi)^{-\nu} \int_{k^2/2 \in [c,d]} \frac{d^\nu k}{e^{\beta(k^2/2-\alpha)}-1}$. (18)

In view of the results we are going to prove with respect to generalized condensation in the van der Waal's limit, it is very important to remark that formulae (7) - (17) actually hold for other free or mean field systems, i.e. for other choices of the one particle Hamiltonian. $F(x)$ will then however no longer be given by expression (6a), but by the $\lim_{L\to\infty} F^L(x)$, where the F^L are as in (6) (see [4] and [5]).

4. The pressure in the van der Waal's limit.

Our first aim is to prove an upper limit for the pressure To do so, we use the lower bound for the interaction term Φ^L derived in [7].

Proposition 1.

$$\lim_{L\to\infty} \sup \; p^L_{int}(\mu) \leq p_a(\mu+b) \tag{19}$$

where $a = \int d^\nu x \, U(x)$ and $b = U(0)$. (20)

Proof: see appendix.

A lower bound can be found by noting that the function

$$f(x) = \frac{1}{\beta L^\nu} \log \mathrm{Tr}\left[\exp\left\{-\beta\left(H^L_a - \mu N^L + x\left(\Phi^L - \frac{a(N^L)^2}{2L^\nu}\right)\right)\right\}\right]$$

is a convex function of x on the interval [0,1] with $f(0) = p^L_a(\mu)$ and $f(1) = p^L_{int}(\mu)$.

This implies that

$$p^L_{int}(\mu) - p^L_a(\mu) \geq f'_+(0) = \frac{a}{2}\,\omega^L_{a,\mu}\left(\left(\frac{N^L}{L^\nu}\right)^2\right) - \omega^L_{a,\mu}\left(\frac{\Phi^L}{L^\nu}\right)$$

(see e.g. [8]).

To state the proposition, we define the functions $F_{\nu/2}(\alpha,x)$ (where $\alpha<0$ if $\nu\leq 2$ and $\alpha\leq 0$ if $\nu\geq 3$) by :

$$F_{\nu/2}(\alpha,x) = (2\pi)^{-\nu}\int d^\nu k \,\frac{e^{ik\cdot x}}{e^{\beta(k^2/2-\alpha)}-1} \quad . \tag{21}$$

Remark that, using (6a) and certain relations for Bessel functions (see [9]), $F_{\nu/2}(\alpha,x)$ can also be written as

$$F_{\nu/2}(\alpha,x) = 2^{\nu/2-1}\,\Gamma(\nu/2)\int dF(u)\,\frac{J_{\nu/2-1}(\sqrt{2u}\,\|x\|)}{(\sqrt{2u}\,\|x\|)^{\nu/2-1}\,(e^{\beta(u-\alpha)}-1)} \quad . \tag{21a}$$

One can then prove

Proposition II.

Let $a = \int d^\nu x\, U(x)$. Then:

1) $\mu < a\rho_c$: $\quad \liminf_{L\to\infty} p^L_{int}(\mu) \geq p_a(\mu) - \frac{1}{2}\int d^\nu x\, U(x)\,(F_{\nu/2-1}(\alpha,x))^2$ (22a)

where α satisfies (11).

2) $\mu \geq a\rho_c$: $\quad \liminf_{L\to\infty} p^L_{int}(\mu) \geq p_a(\mu) - (\rho_a(\mu)-\rho_c)\int d^\nu x\, U(x)\, F_{\nu/2-1}(0,x)$
$- \frac{1}{2}\int d^\nu x\, U(x)\,(F_{\nu/2-1}(0,x))^2$. (22b)

Proof: see appendix .

Combining the bounds of Propositions I and II, the following theorem now follows immediately.

Theorem I.

With the assumptions and notations specified above, one has

i) $\lim_{\lambda\downarrow 0}\ \liminf_{L\to\infty}\ (\limsup_{L\to\infty})\ p^L_{int,\lambda}(\mu) = p_a(\mu)$. (23)

ii) Moreover, if $\rho^L_{int,\lambda}(\mu) \stackrel{def}{=} \omega^L_{int,\lambda,\mu}\left(\frac{N^L}{L^\nu}\right)$

then $\lim_{\lambda\downarrow 0}\ \liminf_{L\to\infty}\ (\limsup_{L\to\infty})\ \rho^L_{int,\lambda}(\mu) = \rho_a(\mu)$. (24)

Proof:

i) From Propositions I and II, we have

$$P_a(\mu) - (\rho_a(\mu) - \rho_c)\,\Theta(\rho_a(\mu) - \rho_c) \int d^\nu x\, U(x)\, F_{\nu/2-1}(\alpha, \tfrac{x}{\lambda}) - \frac{1}{2} \int d^\nu x\, U(x) \left(F_{\frac{\nu}{2}-1}(\alpha, \tfrac{x}{\lambda})\right)^2 \tag{25}$$

$$\leq \liminf_{L\to\infty} \left(\limsup_{L\to\infty}\right) \cdot p^L_{int,\lambda}(\mu) \leq P_a(\mu + \lambda^\nu b),$$

(α as in (11)).

Now note the following facts .

a) $F_{\nu/2}(\mu, x)$ is a bounded function; indeed $|F_{\nu/2}(\mu, x)| \leq \rho_0(\mu)$.

b) $U(x)$ is in $L^1(\mathbb{R}^\nu)$.

We can therefore apply the dominated convergence theorem to the left-hand side of (25) noting that

$$\lim_{\lambda \downarrow 0} F_{\nu/2}(\mu, \tfrac{x}{\lambda}) = 0 \quad , \quad x \neq 0 .$$

This is a direct consequence of (21), the Riemann-Lesbesgue theorem and the fact that

$$k \to \frac{1}{e^{\beta(k^2/2 - \mu)} - 1} \quad \text{is in} \quad L^1(\mathbb{R}^\nu) \quad (\mu < 0 \text{ if } \nu \leq 2,\ \mu \leq 0 \text{ if } \nu > 2)$$

In fact, as $\|x\| \to \infty$, $F_{\nu/2}(\mu, x)$ decreases faster than any polynomial in x if $\mu < 0$ and decreases like $\frac{1}{\|x\|}$ if $\mu = 0$ $(\nu \geq 3)$; see [10].

Moreover $P_a(\mu)$ is a continuous function of μ .

i) now follows immediately.

For ii) we use the Griffith's inequality in the following form (see [11]):

Suppose that $f_{n,m}(x)$ are differentiable, convex functions on the interval $[a-\epsilon, a+\epsilon]$ and suppose that

$$\lim_{n\to\infty} \liminf_{m\to\infty} \left(\limsup_{m\to\infty}\right) f_{n,m}(x) = f(x) .$$

Then $f'_-(a) \leq \lim_{n\to\infty} \liminf_{m\to\infty} (\limsup_{m\to\infty}) f'_{n,m}(a) \leq f'_+(a)$. (26)

The result now follows by noting that

a) $p^L_{int,\lambda}(\mu)$ are convex functions of μ.

b) $\frac{d}{d\mu} p^L_{int,\lambda}(\mu) = \rho^L_{int,\lambda}(\mu)$.

c) $\lim_{\lambda\downarrow 0} \liminf_{L\to\infty} (\limsup_{L\to\infty}) p^L_{int,\lambda}(\mu) = p_a(\mu)$, (see (23)).

A similar result can be proved, when one starts with Neumann boundary conditions instead of periodic boundary conditions. The result of Proposition I clearly remains true, while, by a careful calculation, it can be shown that Proposition II also remains valid. In fact, this can also be seen by noting that $h^L_{0,N} = -\frac{1}{2}\Delta_N \leq h^L_0$, where Δ_N denotes the Laplacian with Neumann boundary conditions [3]). However, repeating the proof of Proposition II yields a weaker lower bound in the case of Dirichlet boundary conditions if $\mu > a\rho_c$. This is due to the fact that, for Dirichlet boundary conditions, the condensate is not spread out uniformly over the box: (see also [6]). Theorem I remains however valid also for these boundary conditions, as can be found in [3].

Energy level occupation densities in the van der Waal's limit.

We now come to the most interesting part of the article, namely, we will now show how the preceding method allows us to find results concerning the number of particles per energy level.

Consider the following Hamiltonians

$$H^L_{a,\sigma} = T^L + \frac{a(N^L)^2}{2L^\nu} + \sigma L^\nu X^L_{[c,d]}$$

$$H^L_{int,\lambda,\sigma} = T^L + \Phi^L_\lambda + \sigma L^\nu X^L_{[c,d]}$$

with $0 < c < d \leq \infty$ and $\sigma > -c$.

These are the Hamiltonians one would get if, instead of starting with the one-particle Hamiltonian h^L, we started with $h^L_\sigma = h^L + \sigma \sum_{\{k: E^L_k \in [c,d]\}} P^L_k$, P^L_k being the projection on the eigenvector ϕ^L_k .

Note that h^L_σ has eigenvalues $E^L_1, \dots, E^L_k, E^L_{k+1}+\sigma, \dots, E^L_{k+n}+\sigma, E^L_{k+n+1}, \dots$,
where $E^L_k < c \leq E^L_{k+1} \leq \cdots \leq E^L_{k+n} \leq d < E^L_{k+n+1} \leq \cdots$.

Define as usual

$$p^L_{a,\sigma}(\mu) = \frac{1}{\beta L^\nu} \log \mathrm{Tr}\,[\exp\{-\beta(H^L_{a,\sigma} - \mu N^L)\}]$$

and similarly $p^L_{int,\lambda,\sigma}(\mu)$. Define also $\rho^L_{a,\sigma}(\mu)$ and $\omega^L_{a,\mu,\sigma}(\cdot)$ in the usual way.

Define moreover $F^L_\sigma(u)$ by

$$\int_{[0,\infty)} f(u)\, dF^L_\sigma(u) = \int_{[c,d]^c} f(u)\, dF^L(u) + \int_{[c,d]} f(u+\sigma)\, dF^L(u).$$

One easily verifies that $\lim_{L\to\infty} F^L_\sigma(u)$ exists: call it $F_\sigma(u)$.

Using standard results of free and mean field systems [4], [5], one then finds:

a) $$p_{0,\sigma}(\mu) = \lim_{L\to\infty} p^L_{0,\sigma}(\mu) = -\frac{1}{\beta} \int_{[0,\infty)} \log\left(1 - \exp\{-\beta(u-\mu)\}\right) dF_\sigma(u). \tag{27}$$

b) $$\rho_{0,\sigma}(\mu) = \lim_{L\to\infty} \rho^L_{0,\sigma}(\mu) = \int_{[0,\infty)} \frac{dF_\sigma(u)}{e^{\beta(u-\mu)} - 1}, \tag{28}$$

c) $$p_{a,\sigma}(\mu) = \lim_{L\to\infty} p^L_{a,\sigma}(\mu) = p_{0,\sigma}(\alpha) + \frac{(\mu-\alpha)^2}{2a}, \quad a > 0, \tag{29}$$

where

$$\alpha = \begin{cases} \mu - a\rho_{0,\sigma}(\alpha) & \text{if } \mu < a\rho_{0,\sigma}(0), \\ 0 & \text{if } \mu \geq a\rho_{0,\sigma}(0). \end{cases} \tag{30}$$

d) $$\rho_{a,\sigma}(\mu) = \lim_{L\to\infty} \rho^L_{a,\sigma}(\mu) = \frac{\mu-\alpha}{a}, \qquad (\alpha \text{ as in } (26)). \tag{31}$$

$$\lim_{L\to\infty} \omega^L_{a,\mu,\sigma}(X^L_{[c,d]}) = \delta_{c,0}(\rho_{a,\sigma}(\mu) - \rho_{0,\sigma}(0))\,\theta(\rho_{a,\sigma}(\mu) - \rho_{0,\sigma}(0)) + \int_{[c,d]} \frac{dF_\sigma(u)}{e^{\beta(u-\alpha)} - 1}. \tag{32}$$

$$\lim_{L\to\infty} \omega^L_{a,\mu,\sigma}(X^L_{[c_1,d_1]} X^L_{[c_2,d_2]}) = \lim_{L\to\infty} \omega^L_{a,\mu,\sigma}(X^L_{[c_1,d_1]}) \lim_{L\to\infty} \omega^L_{a,\mu,\sigma}(X^L_{[c_2,d_2]}). \tag{33}$$

Moreover let

$$F^{\sigma}_{\nu/2}(\mu,x) = (2\pi)^{-\nu}\int_{\frac{k^2}{2}\in[c,d]^c} d^\nu k \, \frac{e^{ik\cdot x}}{e^{\beta(k^2/2-\mu)}-1}$$

$$+ (2\pi)^{-\nu}\int_{\frac{k^2}{2}\in[c,d]} d^\nu k \, \frac{}{e^{\beta(k^2/2+\sigma-\mu)}-1} \quad , \qquad (34)$$

$(\mu<0$ if $\nu\leq 2$, $\mu\leq 0$ if $\nu\geq 3)$.

Repeating the arguments of Proposition I and II, we easily find

Proposition III.

i) $\limsup_{L\to\infty} p^L_{int,\lambda,\sigma}(\mu) \leq p^\sigma_a(\mu+\lambda^\nu b)$

ii) $\liminf_{L\to\infty} p^L_{int,\lambda,\sigma}(\mu) \geq p^\sigma_a(\mu) - (\rho_{a,\sigma}(\mu)-\rho_{c,\sigma})\theta(\rho_{a,\sigma}(\mu)-\rho_{c,\sigma})\int d^\nu x\, U(x) F^\sigma_{\nu/2}(\alpha,\frac{x}{\lambda})$

$$-\frac{1}{2}\int d^\nu x\, U(x)\left(F^\sigma_{\nu/2}(\alpha,\frac{x}{\lambda})\right)^2 ,$$

with α satisfying (30) and $\rho_{c,\sigma} \stackrel{def}{=} \rho_{0,\sigma}(0)$.

Note that, as with $F_{\nu/2}(\mu,x)$, so $F^\sigma_{\nu/2}(\mu,x)$ has the properties:

i) $F^\sigma_{\nu/2}(\mu,x)$ is bounded;

ii) $\lim_{\lambda\downarrow 0} F^\sigma_{\nu/2}(\mu,\frac{x}{\lambda}) = 0$ if $x \neq 0$.

This implies

Theorem II.

$$\lim_{\lambda\downarrow 0}\ \liminf_{L\to\infty}\ (\limsup_{L\to\infty})\ p^L_{int,\lambda,\sigma}(\mu) = p_{a,\sigma}(\mu) . \qquad (35)$$

We are now able to prove the most interesting result of this paper.

Theorem III.

For the model described in §2, one has

i) $$\lim_{\lambda\downarrow 0}\ \liminf_{L\to\infty}\ (\limsup_{L\to\infty})\ \omega^L_{int,\lambda,\mu}(X^L_{[c,d]}) = \lim_{L\to\infty}\omega_{a,\mu}(X^L_{[c,d]}) , \quad 0\leq c<d\leq\infty, \qquad (36)$$

ii) If $\mu > a\rho_c$ then

$$\lim_{\epsilon \downarrow 0} \lim_{\lambda \downarrow 0} \liminf_{L\to\infty} (\limsup_{L\to\infty}) \; \omega^L_{int,\lambda,\mu}(X^L_{[0,\epsilon]}) = \rho_a(\mu) - \rho_c \, . \tag{37}$$

Proof:

i) Let us consider the case where c=0 (the case c≠0, has to be treated separately, but is essentially the same).

Consider the function

$$\pi^L_\lambda(\sigma) = \frac{1}{\beta L^\nu} \log \mathrm{Tr}\left[\exp\{-\beta(H^L_{int,\lambda} - \mu N^L - \sigma L^\nu X^L_{[0,\epsilon]})\}\right]$$

$$= \frac{1}{\beta L^\nu} \log \mathrm{Tr}\left[\exp\{-\beta(H^L_{int,\lambda} - (\mu+\sigma)N^L + \sigma L^\nu X^L_{(\epsilon,\infty)})\}\right] .$$

From Theorem II, it follows that

$$\lim_{\lambda\downarrow 0} \liminf_{L\to\infty} (\limsup_{L\to\infty}) \; \pi^L_\lambda(\sigma) = p_{a,\sigma}(\mu+\sigma) = p_{0,\sigma}(\alpha) + \frac{(\mu+\sigma-\alpha)^2}{2a}$$

where α satisfies

$$\alpha = \begin{cases} \mu+\sigma - a\, p_{0,\sigma}'(\alpha) \, , & \mu+\sigma \leq a\rho_{c,\sigma} \, , \\ 0 \, , & \mu+\sigma > a\rho_{c,\sigma} \, . \end{cases}$$

Now, for $\lambda > 0, L > 0$, $\pi^L_\lambda(\sigma)$ is a convex function of σ on the interval $[-\epsilon,\infty]$ and

$$\frac{d}{d\sigma}\pi^L_\lambda(\sigma)\Big|_{\sigma=0} = \omega^L_{int,\lambda,\mu}(X^L_{[0,\epsilon]})$$

(see e.g. [8]).

Applying the Griffith's lemma in the form (26), one finds consequently:

$$f'_-(0) \leq \lim_{\lambda\downarrow 0} \liminf_{L\to\infty} (\limsup_{L\to\infty}) \; \omega^L_{int,\lambda,\mu}(X^L_{[0,\epsilon]}) \leq f'_+(0) \, , \tag{38}$$

where

$$f(\sigma) = \rho_{a,\sigma}(\mu+\sigma) \, .$$

Using formulae (27) - (29), it can now easily be shown that f is differentiable and that

a) $\mu < a\rho_c$: $$f'(0) = \frac{\mu - \alpha_0}{a} - \int_{[\epsilon,\infty)} \frac{dF(u)}{e^{\beta(u-\alpha_0)}-1} ,$$

where α_0 satisfies:

$$\mu - \alpha_0 = a\rho_0(\alpha_0) = a \int_{[0,\infty)} \frac{dF(u)}{e^{\beta(\mu-\alpha_0)}-1} ;$$

therefore:

$$f'(0) = \int_{[0,\epsilon)} \frac{dF(u)}{e^{\beta(u-\alpha_0)}-1} = \lim_{L\to\infty} \omega^L_{a,\mu}(X^L_{[0,\epsilon)}) . \tag{39a}$$

b) $\mu > a\rho_c$: $$f'(0) = \frac{\mu}{a} - \int_{(\epsilon,\infty)} \frac{dF(u)}{e^{\beta u}-1} ;$$

note that $\mu = a\rho_a(\mu)$ and that $\rho_c = \int_{[0,\infty)} \frac{dF(u)}{e^{\beta u}-1}$;

hence

$$f'(0) = \rho_a(\mu) - \rho_c + \int_{[0,\epsilon)} \frac{dF(u)}{e^{\beta u}-1} = \lim_{L\to\infty} \omega^L_{a,\mu}(X^L_{[0,\epsilon)}) . \tag{39b}$$

c) $\mu = a\rho_c$: in this case, the result follows from comparing right and left-hand derivatives separately and seeing that they are equal to $\lim\limits_{L\to\infty} \omega^L_{a,\mu}(X^L_{[0,\epsilon)})$. (39c)

Combining (38) and (39a-c), expression (36) follows.

d) is now a direct consequent of (36).

We have therefore shown that generalized condensation persists in the van der Waal's limit. This result might not seem very surprising and it is clear that a result like

$$\lim_{\lambda\downarrow 0} \lim_{\epsilon\downarrow 0} \liminf_{L\to\infty} (\limsup_{L\to\infty}) \; \omega^L_{int,\lambda,\mu}(X^L_{[0,\epsilon)}) = \rho_a(\mu) - \rho_c$$

(for $\rho > a\rho_c$), would be much more interesting, as it would imply that for λ small enough the interacting system has a condensate. However, it is, as far as we know, one of the few rigourous results about the presence of condensation in the interacting system, even in the van der Waal's limit. Recent studies [12], [13] show how easily the phenomenon of Bose-Einstein condensation is destroyed when introducing an interaction, which is however gentle enough to preserve the singularity in the thermodynamic function. To put it differently: the fact that $\lim_{\lambda\downarrow 0}$ $\liminf_{L\to\infty}$ $(\limsup_{L\to\infty})$ $p^L_{int,\lambda}(\mu)$ has a singularity at $\mu = a\rho_c$, is in itself in no way evidence that there will be condensation (see [12], [13]). As far as we know, there is only one other rigourous result concerning the presence of Bose-Einstein condensation: namely, if the Hamiltonian of the free gas has a gap in its spectrum, then Bose-Einstein condensation is stable under perturbation by any integrable two-body potential of positive type [14]. In that article, the result was proved using the Bogoliubov approximation. It can also be proved using arguments similar to those presented in this article.

Appendix: Proofs of Proposition I and II.

Proof of Proposition I.

In a previous paper [7], we proved the following inequality for a potential Φ^L satisfying conditions (1) - (4) :
given $\epsilon > 0$ there exists L_0 such that for $L > L_0$, $n > 0$ and x_j in Λ^L

$$\Phi(x_1,\dots,x_n) \stackrel{\text{def}}{=} \sum_{1\le i<j\le n} U(x_i - x_j) \geq \frac{a(1-\epsilon)n^2}{2L^\nu} - bn ,$$

where $a = \int d^\nu x \, U(x)$ and $b = U(0)$.

(Remark: the conditions on the regions Λ^L, specified in [7], are satisfied when the Λ^L are cubes).

This implies that

$$H^L_{int} - \mu N^L \geq H^L_a + \frac{a(1-\epsilon)(N^L)^2}{2L^\nu} - (\mu + b) N^L ,$$

or $p^L_{int}(\mu) \leq p^L_{a(1-\epsilon)}(\mu+b)$, (as $A \geq B$ implies $\mathrm{Tr}\exp\{-A\} \leq \mathrm{Tr}\exp\{-B\}$).

The result follows by taking the $\limsup_{L\to\infty}$ and by letting ϵ tend to zero.

<u>Proof of Proposition II.</u>

Define the function f(x) by

$$f(x) = \frac{1}{\beta L^\nu} \log \mathrm{Tr}\left[\exp\left\{-\beta\left(H^L_a - \mu N^L + x\left(\Phi^L - \frac{a(N^L)^2}{2L^\nu}\right)\right)\right\}\right] .$$

f(x) is a convex function of x on the interval [0,1] with $f(0) = p^L_a(\mu)$ and $f(1) = p^L_{int}(\mu)$.

As f is convex, this implies

$$p^L_{int}(\mu) - p^L_a(\mu) \geq f'_+(0) = \omega^L_{a,\mu}\left(\frac{\Phi^L}{L^\nu}\right) - \frac{a}{2}\, \omega^L_{a,\mu}\left(\left(\frac{N^L}{L^\nu}\right)^2\right) , \tag{41}$$

(see e.g. [9]).

Now

$$\begin{aligned}\omega^L_{a,\mu}\left(\frac{\Phi^L}{L^\nu}\right) &= \frac{1}{2L^\nu}\int d^\nu x \int d^\nu y\, U(x-y)\, \omega^L_{a,\mu}\left(a^*(x)a^*(y)a(y)a(x)\right) \\ &= \frac{1}{2L^\nu}\int d^\nu x \int d^\nu y\, U(x-y) \sum_{k_1,k_2,k_3,k_4} \bar f_{k_1}(x) \bar f_{k_2}(y) f_{k_3}(y) f_{k_4}(x)\, \omega^L_{a,\mu}\left(a^*_{k_1} a^*_{k_2} a_{k_3} a_{k_4}\right) ,\end{aligned} \tag{42}$$

where $a^{(*)}_k$ stands as usual for $a^{(*)}(f^L_k)$.

A simple inspection of the state $\omega^L_{a,\mu}$, however, permits us to simplify this expression, as

$$^L(a^*_{k_1}\ a^*_{k_2}\ a_{k_3}\ a_k\ ) = 0 \quad \text{unless} \quad k_1 = k_3 \text{ and } k_2 = k_4$$
$$\text{or} \quad k_1 = k_4 \text{ and } k_2 = k_3 \ .$$

Therefore expression (42) reduces to (note that $N^L_k = a^*_k\, a_k$)

$$\omega^L_{a,\mu}\left(\frac{\Phi^L}{L^\nu}\right) = \frac{1}{2L^\nu}\int d^\nu x\int d^\nu y\, U(x-y)\Big\{\sum_{k_1,k_2} |f^L_{k_1}(x)|^2 |f^L_{k_2}(y)|^2\, \omega^L_{a,\mu}(N^L_{k_1} N^L_{k_2})$$
$$+\sum_{k_1} |f^L_{k_1}(x)|^2 |f^L_{k_2}(y)|^2\, \omega^L_{a,\mu}((N^L_{k_1})^2) + \sum_{k_1\neq k_2} \bar f^L_{k_1}(x)\, \bar f^L_{k_2}(y)\, f^L_{k_2}(y)\, f^L_{k_1}(x)\, \omega^L_{a,\mu}(N^L_{k_1} N^L_{k_2})\Big\} . \tag{43}$$

Now note that the eigenfunctions $f^L_k(x)$ are of the form $L^{-\nu/2}\exp(i\xi\cdot x)$ with $\xi = \left(\frac{2\pi n_1}{L}, \ldots, \frac{2\pi n_\nu}{L}\right), n_j \in \mathbb{Z}$ and write, for convenience, N^L instead of N^L_k.

(43) then reads:

$$\omega^L_{a,\mu}\left(\frac{\Phi^L}{L^\nu}\right) = \frac{1}{2L^\nu}\int d^\nu x\int d^\nu y\, U(x-y)\left[\omega^L_{a,\mu}\left(\left(\frac{N^L}{L^\nu}\right)^2\right) + \frac{1}{L^\nu}\,\omega^L_{a,\mu}\left(\frac{N^L}{L^\nu}\right)\right]$$
$$+\frac{1}{2L^\nu}\int d^\nu x\int d^\nu y\, U(x-y)\sum_{\xi_1,\xi_2} e^{i(\xi_1-\xi_2)\cdot(x-y)}\,\omega^L_{a,\mu}(N^L_{\xi_1} N^L_{\xi_2}) . \tag{44}$$

To prove the proposition, one now has to observe the folowing facts.

1) Given $\epsilon > 0$, $\exists L_0$ such that for $L > L_0$ and for any bounded function f on $\mathbb{R}^\nu$,

$$\left| \frac{1}{L^\nu} \int_{\Lambda^L} d^\nu x \int_{\Lambda^L} d^\nu y\, U(x-y) f(x-y) - \int_{\mathbb{R}^\nu} d^\nu u\, U(u) f(u) \right| < \epsilon A ,$$

where $A = \max_{x \in \mathbb{R}^\nu} |f(x)|$. (For a proof of this result, see [7]).

2) Define $X^L_{[c,d]}$ as in §3.

Then $$\lim_{L\to\infty} \omega^L_{a,\mu}(X^L_{[c_1,d_1]} X^L_{[c_2,d_2]}) = \lim_{L\to\infty} \omega^L_{a,\mu}(X^L_{[c_1,d_1]}) \lim_{L\to\infty} \omega^L_{a,\mu}(X^L_{[c_2,d_2]}) ,$$

(see (17)).

3) $$\lim_{L\to\infty} \omega^L_{a,\mu}(X^L_{[c,d]}) = \delta_{c,0}(\rho_a(\mu) - \rho_c)\theta(\rho_a(\mu) - \rho_c)$$

$$+ (2\pi)^{-\nu} \int_{k^2/2 \in [c,d]} \frac{d^\nu k}{e^{\beta(k^2/2 - \alpha)} - 1}$$

where $$\alpha = \begin{cases} \mu - a\rho_0(\alpha) , & \text{if } \mu < a\rho_c , \\ 0 , & \text{if } \mu \geq a\rho_c , \end{cases}$$

(see (16) and (18)).

Noting these facts and combining (41) and (44), the statement of the proposition then follows after a straightforward calculation.

Acknowledgements: One of us (Ph. d. S.) would like to thank the N.F.W.O. and D.I.A.S. for financial support.

References.

[1] Kac M, Uhlenbeck G., Hemmer P.: J. Math. Phys. 4, 216 (1963).

[2] Lebowitz J., Penrose O., J. Math. Phys. 7, 98 (1966).

[3] Lieb E.: J. Math. Phys. 7, 1016 (1966).

[4] van den Berg M., Lewis J.T., Pulè J.V.: General Theory of Bose-Einstein condensation (to appear in Helv. Phys. Acta).

[5] van den Berg M., Lewis J.T., de Smedt Ph.: J. Stat. Phys. 37, 697 (1984).

[6] de Smedt Ph.: Ph.D. thesis, K.U. Leuven (1983).

[7] Lewis J.T., Pulè J.V., de Smedt Ph.: J. Stat. Phys. 35, 381 (1984).

[8] Ginibre J.: Comm. Math. Phys. 8, 26 (1968).

[9] Schwartz L.: Mathematics for the physical sciences Addison-Wesley: 1966.

[10] Buckingham M.J., Gunton J.D.: Phys. Rev. 166, 152 (1968).

[11] Hepp K., Lieb E.: Phys. Rev. A8 2517 (1973).

[12] Buffet E., Pulè J.V.: A hard core Bose gas: accepted for publication in J. Stat. Phys. .

[13] Buffet E., Pulè J.V.: Hard Bosons in One Dimension: submitted to Comm. Math. Phy

[14] Buffet E., de Smedt Ph., Pulè J.V.: J. Phys. A, 16, 4307 (1983).